A CHARACTERIZATION OF MULTIPLIER SEQUENCES FOR GENERALIZED LAGUERRE BASES

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Abstract. We give a complete characterization of multiplier sequences for generalized Laguerre bases. We also apply our methods to give a short proof of the characterization of Hermite multiplier sequences achieved by Piotrowski.

1. Introduction

In this paper we study linear operators on real polynomials that preserve the property of having only real zeros (we consider constant polynomials as being real–rooted). Pólya and Schur characterized such linear operators that act diagonally on the standard basis of $\mathbb{R}[x]$, see [14]. A complete characterization of linear operators preserving real–rootedness was achieved only recently by Borcea and the first author in [3]. However, generalizations of the Pólya–Schur theorem of the following form are still open in many important cases:

Problem 1. *Let $\mathscr{P} = \{P_n(x)\}_{n=0}^{\infty}$ be sequence of real polynomials. For a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of real numbers, define a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ by*

$$T(P_n(x)) = \lambda_n P_n(x), \qquad \text{for all } n \in \mathbb{N} := \{0, 1, 2, \dots\}.$$

Characterize the sequences $\{\lambda_n\}_{n=0}^{\infty}$ for which T preserves real–rootedness.

We call such a sequence a *$\mathscr{P}$-multiplier sequence*, while the term *multiplier sequence* is reserved for the classical case $\mathscr{P} = \{x^n\}_{n=0}^{\infty}$. The case of Problem 1 when $\mathscr{P} = \{x^n\}_{n=0}^{\infty}$ goes back to Laguerre and Jensen and was completely solved by Pólya and Schur in [14], see also [7, 12]. Turán [17] and subsequently Bleecker and Csordas [2] provided classes of multiplier sequences for the Hermite polynomials $\mathscr{H} = \{H_n(x)\}_{n=0}^{\infty}$, while Piotrowski completely characterized $\mathscr{H}$-multiplier sequences in [13]. Recently partial results regarding multiplier sequences for the generalized Laguerre bases [9], and for the Legendre bases [1], were achieved.

Recall that the *(generalized) Laguerre polynomials*, $\mathscr{L}_\alpha = \{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$, are defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad \alpha > -1, \tag{1.1}$$

see [16]. In this paper we give a complete characterization of $\mathscr{L}_\alpha$-multiplier sequences for each $\alpha > -1$. We say that a sequence $\{\lambda_n\}_{n=0}^{\infty}$ is trivial if there is a number $k \in \mathbb{N}$ such that $\lambda_n = 0$ for all $n \notin \{k, k+1\}$. It is not hard to see that all trivial sequences are $\mathscr{L}_\alpha$-multiplier sequences, see [9, Proposition 2.1]. Hence it

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remains to characterize non-trivial $\mathscr{L}_\alpha$-multiplier sequences, which is achieved by the following:

Theorem 1.1. *Let $p(y) = \sum_{k=0}^{\infty} \binom{k+\alpha}{k} a_k y^k$ be a formal power series where $\alpha > -1$, and let $\{\lambda_n\}_{n=0}^{\infty}$ be a non-trivial sequence defined by*

$$\lambda_n := \sum_{k=0}^{n} a_k \binom{n}{k}.$$

Then $\{\lambda_n\}_{n=0}^{\infty}$ is an $\mathscr{L}_\alpha$-multiplier sequence if and only if $p(y)$ is a real-rooted polynomial with all its zeros contained in the interval $[-1, 0]$.

Remark 1.2. Note that Theorem 1.1 implies that each non-trivial $\mathscr{L}_\alpha$-multiplier sequence is a polynomial in n, and hence that the corresponding operator T is a finite order differential operator.

Remark 1.3. We may express an arbitrary sequence $\{\lambda_n\}_{n=0}^{\infty}$ as

$$\lambda_n = \sum_{k=0}^{n} a_k \binom{n}{k}, \quad \text{where} \quad a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_k.$$

It follows (by elementary binomial identities) that the series $p(y)$, defined in Theorem 1.1, may be expressed in terms of the sequence $\{\lambda_n\}_{n=0}^{\infty}$ as the formal power series

$$p(y) = \frac{1}{(1+y)^{\alpha+1}} \sum_{n=0}^{\infty} \lambda_n \binom{n+\alpha}{n} \left(\frac{y}{1+y}\right)^n. \tag{1.2}$$

Hence $\{\lambda_n\}_{n=0}^{\infty}$ is a non-trivial $\mathscr{L}_\alpha$-multiplier sequence if and only if (1.2) is a real-rooted polynomial with all its zeros contained in the interval $[-1, 0]$.

Our method of proving Theorem 1.1 is applicable to other bases, and in Section 3 we give a short proof of the characterization of Hermite-multiplier sequences due to Piotrowski [13].

2. Proof of Theorem 1.1

The main tool used to prove Theorem 1.1 is the characterization of linear preservers of real–rotedness achieved in [3], which we now describe. The *symbol* of a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is the formal power series defined by

$$G_T(x, y) := \sum_{n=0}^{\infty} \frac{(-1)^n T(x^n)}{n!} y^n.$$

The *Laguerre–Pólya class*, $\mathscr{L}\text{-}\mathscr{P}_1(\mathbb{R})$, consists of all real entire functions that are limits, uniformly on compact subsets of $\mathbb{C}$, of real–rooted polynomials. Laguerre and Pólya proved that an entire function Φ is in the Laguerre–Pólya class if and only it may be expressed in the form

$$\Phi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k}, \quad \omega \in \mathbb{N} \cup \{\infty\}, \tag{2.1}$$

where $c, \beta, x_k \in \mathbb{R}$ for all k, $c \neq 0, \alpha \geq 0$, n is a non-negative integer and $\sum_{k=1}^{\infty} x_k^{-2} < \infty$. A multivariate polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ is called *stable* if $P(x_1, \ldots, x_n) \neq 0$ whenever $\mathrm{Im}(x_j) > 0$ for all $1 \leq j \leq n$. Hence a real univariate polynomial is stable if and only if it is real–rooted. The *Laguerre–Pólya class* of

real entire functions in n variables, $\mathscr{L}\text{-}\mathscr{P}_n(\mathbb{R})$, consists of all real entire functions in that are limits, uniformly on compact subsets of $\mathbb{C}$, of real stable polynomials.

Theorem 2.1 ([3]). *A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ preserves real–rootedness if and only if*

(1) *The rank of T is at most two and T is of the form*
$$T(P) = \alpha(P)Q + \beta(P)R,$$
where $\alpha, \beta : \mathbb{R}[x] \to \mathbb{R}$ are linear functional and $Q + iR$ is a stable polynomial, or;
(2) $G_T(x, y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$*, or;*
(3) $G_T(-x, y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$*.*

Theorem 2.1 suggest that we should find necessary and sufficient conditions for the symbol, $G_T(x, y)$, of the operator given by $T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)$ to be in $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$. We shall need an expression for $G_T(x, y)$. Lemma 2.2 follows from [9, Proposition 3.2], but we give here a proof based on a well known identity for Laguerre polynomials.

Lemma 2.2. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers. The symbol of the operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(L_n^{\alpha}(x)) = \lambda_n L_n^{\alpha}(x)$, for all $n \in \mathbb{N}$, is given by*
$$G_T(x, y) = e^{-xy} \sum_{n=0}^{\infty} a_n x^n L_n^{(\alpha)}(xy + x).$$
where
$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_k, \quad n \in \mathbb{N}.$$

Proof. Recall the differential equation satisfied by the Laguerre polynomials:
$$nL_n^{\alpha}(x) = (x - \alpha - 1)\frac{d}{dx}L_n^{\alpha}(x) - x\frac{d^2}{dx^2}L_n^{\alpha}(x),$$
see [16]. Consider the operator $\delta := (x - \alpha - 1)d/dx - xd^2/dx^2$ and let
$$\binom{\delta}{k} := \frac{\delta(\delta - 1)\cdots(\delta - k + 1)}{k!}.$$
Then $\binom{\delta}{k} L_n^{(\alpha)}(x) = \binom{n}{k} L_n^{(\alpha)}(x)$, and letting T be the operator corresponding to $\{\lambda_n\}_{n=0}^{\infty}$, we have $T = \sum_{k=0}^{\infty} a_k \binom{\delta}{k}$. Let S_k denote the operator $\binom{\delta}{k}$. Then, by the change of variables $y = t/(t - 1)$, in the generating function for the Laguerre polynomials:
$$\frac{e^{-xt/(1-t)}}{(1 - t)^{1+\alpha}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n,$$
see [16], yields
$$G_{S_k}(x, y) = S_k(e^{-xy}) = \sum_{n=0}^{\infty} \binom{n}{k} L_n^{(\alpha)}(x)y^n(1 + y)^{-n-\alpha-1}.$$

On the other hand, with the same change of variables as above, identity (9) on page 211 in [16] states that

$$\sum_{n=0}^{\infty} \binom{n}{k} L_n^{(\alpha)}(x) y^n (1+y)^{-n-1-\alpha} = e^{-xy} \sum_{n=0}^{\infty} a_n y^n L_n^{(\alpha)}(xy+x),$$

from which the lemma follows by linearity. $\square$

The explicit expression (1.1) of the Laguerre polynomials now yields:

$$G_T(x,y) = e^{-xy} \sum_{n=0}^{\infty} a_n y^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x(y+1))^k}{k!}, \tag{2.2}$$

and since

$$\frac{p^{(k)}(y)}{(\alpha+1)\cdots(\alpha+k)} = \sum_{n=k}^{\infty} \binom{n+\alpha}{n-k} a_n y^{n-k},$$

where $p(y) = \sum_{n=0}^{\infty} \binom{n+\alpha}{n} a_n y^n$, changing the order of summation in (2.2) yields the following consequence of Lemma 2.2:

Corollary 2.3. *The symbol of the operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(L_n^{\alpha}(x)) = \lambda_n L_n^{\alpha}(x)$, for all $n \in bN$, is given by*

$$G_T(x,y) = e^{-xy} \sum_{k=0}^{\infty} p^{(k)}(y) \frac{(-xy(y+1))^k}{(\alpha+1)\cdots(\alpha+k)k!},$$

where $p(y)$ is defined as in Theorem 1.1.

Before we proceed with the proof of Theorem 1.1 let us collect some fundamental properties of multiplier sequences in a lemma for ease of reference:

Lemma 2.4.

(1) *(Pólya and Schur, [14]). Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers, and define a formal power series by*

$$\Phi(x) := \sum_{k=0}^{\infty} \lambda_k \frac{x^k}{k!}.$$

Then $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier sequence if and only if $\Phi(x)$ or $\Phi(-x)$ is an entire function that has the form

$$cx^n e^{sx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{\alpha_k}\right), \quad \omega \in \mathbb{N} \cup \{\infty\}, \tag{2.3}$$

where $s \geq 0$, $n \in \mathbb{N}$, $c \neq 0$, $\alpha_k > 0$ for all k, and $\sum_{k=0}^{\infty} \alpha_k^{-1} < \infty$.

(2) *If $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier sequence and $\lambda_k \lambda_\ell \neq 0$ for some $k < \ell$, then $\lambda_i \neq 0$, for all $k \leq i \leq \ell$. This follows easily from (2.3).*

(3) *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers and let T be corresponding diagonal operator. Then T has rank at most two and $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier sequence if and only if $\{\lambda_n\}_{n=0}^{\infty}$ is a trivial sequence (as defined in the introduction). This follows easily from (2) above.*

2.1. **Proof of Necessity.** Any $\mathscr{L}_\alpha$-multiplier sequence is a multiplier sequence, see [13, Lemma 157]. Assume that $\{\lambda_n\}_{n=0}^\infty$ is a non-trivial $\mathscr{L}_\alpha$-multiplier sequence, and let T be the corresponding operator. Then, by Theorem 2.1, $G_T(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ or $G_T(-x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$, since if T has rank at most two then $\{\lambda_n\}_{n=0}^\infty$ is trivial by Lemma 2.4 (3). Assume $G_T(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ and expand the expression of $G_T(x,y)$ in Corollary 2.3 in powers of x:

$$G_T(x,y) = p(y) - x\left(yp(y) + \frac{y(y+1)}{1+\alpha}p'(y)\right) + \cdots$$

Non-negative multiplier sequences may be extended to act on functions of two variables by the rule $x^k y^\ell \mapsto \lambda_k x^k y^\ell$ for all $k, \ell \in \mathbb{N}$. The class $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ is preserved under this action (see [6] and Lemma 3.7 of [4]). Hence we may truncate the expression above by the multiplier sequence $\{1,0,0,...\}$ and obtain $p(y) \in \mathscr{L}\text{-}\mathscr{P}_1(\mathbb{R})$. If we instead truncate by the multiplier sequence $\{1,1,0,0,\dots\}$ we arrive at the bivariate expression

$$Q(x,y) = p(y) - x\left(yp(y) + \frac{y(1+y)}{(1+\alpha)}p'(y)\right) \tag{2.4}$$

which belongs to $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$.

To conclude something from this we shall need a version of the Hermite–Biehler theorem and the notions of *interlacing zeros* and *proper position*. Let $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_m$ be the zeros of two real–rooted polynomials f and g, where $\deg f = n$, $\deg g = m$ and $|n-m| \le 1$. We say that the zeros of f and g interlace if they can be ordered so that $x_1 \le y_1 \le x_2 \le y_2 \le \cdots$ or $y_1 \le x_1 \le y_2 \le x_2 \le \cdots$ If the zeros of two polynomials f and g interlace, then the *Wronskian*

$$W[f,g] := f'g - fg'$$

is either non-negative or non-positive on the whole of $\mathbb{R}$. In the case when $W[f,g] \le 0$ we say that f and g are in *proper position*, and we denote this by $f \ll g$.

Theorem 2.5 (Hermite-Biehler, see e.g. [15]). *Let $f, g \in \mathbb{R}[x]$, not both identically zero. Then $f \ll g$ if and only if the polynomial $g + if$ is stable.*

We may extend the the notion of proper position to $\mathscr{L}\text{-}\mathscr{P}_1(\mathbb{R})$ by setting $f \ll g$ if and only if $g + if \in \mathscr{L}\text{-}\mathscr{P}_1(\mathbb{C})$, where $\mathscr{L}\text{-}\mathscr{P}_1(\mathbb{C})$ is the *complex Laguerre-Pólya class* which is defined to be the set of entire functions that are limits, uniformly on compact subsets of $\mathbb{C}$, of stable polynomials in $\mathbb{C}[x]$. In particular if $f \ll g$, then $W[f,g](x) \le 0$ for all $x \in \mathbb{R}$.

Consider $Q(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ from (2.4) and set $q(y) = yp(y) + y(1+y)p'(y)/(1+\alpha)$. Then $iQ(i,y) = q(y) + ip(y) \in \mathscr{L}\text{-}\mathscr{P}_1(\mathbb{C})$, and thus

$$W[p,q](y) = -p^2(y) + \frac{y(y+1)}{1+\alpha}((p'(y))^2 - p(y)p(y)'') - \frac{2y+1}{1+\alpha}p(y)p'(y) \le 0,$$

for all $y \in \mathbb{R}$. Evaluating $W[p,q]$ at a simple zero y_0 of $p(y)$ yields $y_0(y_0+1)(p'(y_0))^2 \le 0$ which can only happen if $y_0 \in [-1,0]$.

For multiple zeros we proceed as follows. Consider again $W[p,q](y)$ and a real zero y_0 of $p(y)$ of multiplicity $M \ge 2$. The multiplicity of y_0 will be $2M$ for p^2, $2M-1$ for pp' and $2M-2$ for $(p')^2$ and pp''. If there is no cancellation the

dominating term near y_0 of $W[p,q](y)$ is

$$\frac{y(y+1)}{\alpha+1}(p'(y)^2 - p(y)p''(y)). \tag{2.5}$$

To see that there is no cancellation we write $(p'(y))^2 - p(y)p''(y) = (y-y_0)^{2M-2}R(y)$, and prove that $R(y_0) > 0$. Write $p(y) = (y-y_0)^M s(y)$ and obtain

$$(p'(y))^2 - p(y)p''(y) = (y-y_0)^{2M-2}\big(Ms(y)^2 + (s'(y)^2 - s(y)s''(y))(y-y_0)^2\big).$$

Now the Laguerre inequality (see e.g. [7, Corollary 3.7]) states that

$$f'(x)^2 - f(x)f''(x) \geq 0, \quad x \in \mathbb{R},$$

for any $f(x) \in \mathscr{L}\text{-}\mathscr{P}_1(\mathbb{R})$. Thus $R(y_0) \geq Ms(y_0)^2 > 0$ which proves that (2.5) is the dominating term near y_0 and from which it follows that $y_0 \in [-1,0]$.

We know that $p(y)$ is an entire function in $\mathscr{L}\text{-}\mathscr{P}_1(\mathbb{R})$ so it has the form (2.1), and we now show that it is in fact a polynomial. Since its zeros lie in the interval $[-1,0]$, it can only have a finite number of zeros, that is, $p(y) = e^{ay-by^2}K(y)$, where $K(y)$ is a real–rooted polynomial with zeros only in $[-1,0]$, and $a,b \in \mathbb{R}$ with $b \geq 0$. Now $Q(x,y) = e^{ay-by^2}(K(y) - xF(y))$ where

$$F(y) = yK(y) + \frac{y(y+1)}{1+\alpha}\big((a-2by)K(y) + K'(y)\big).$$

The zeros of $F(y)$ and $K(y)$ interlace by Theorem 2.5 (set $x = i$). Notice that $\deg F \geq \deg K + 2$, unless $a = b = 0$. Hence $a = b = 0$ and there is no exponential factor. This completes the proof that $p(y)$ is a real–rooted polynomial with all its zeros contained in $[-1,0]$, and finishes the proof of necessity in the case when $G_T(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$. It remains to prove that we cannot have $G_T(-x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$.

Assume $G_T(-x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$. Then proceeding as for the case when $G_T(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$, we get $q(y) \ll p(y)$ where $q(y)$ is defined as above. Thus

$$W[p,q](y) = -p^2(y) + \frac{y(y+1)}{1+\alpha}((p'(y))^2 - p(y)p(y)'') - \frac{2y+1}{1+\alpha}p(y)p'(y) \geq 0, \tag{2.6}$$

for all $y \in \mathbb{R}$. If $p(-1/2) \neq 0$, then Laguerre's inequality implies that the middle term in (2.6) is non-positive and thus $W[p,q](-1/2) < 0$. Suppose $y = -1/2$ is a zero of $p(y)$ of multiplicity $M \geq 1$. Then, since

$$(y+1/2)\frac{p'(y)}{p(y)} \approx M,$$

near $y = -1/2$ we see that also the last term in (2.6) is negative near $y = -1/2$. Hence we cannot have $G_T(-x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$.

2.2. **Proof of Sufficiency.** We now prove that the conditions on $p(y)$ in Theorem 1.1 imply $G_T(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$, which will then prove sufficiency by Theorem 2.1. Assume that the zeros of

$$p(y) = \sum_{k=0}^{n} \binom{k+\alpha}{k} a_k y^k = \prod_{j=0}^{n}(y+\theta_j)$$

are real and lie in $[-1, 0]$, and consider again the symbol expressed as in Corollary 2.3:

$$G_T(x,y) = e^{-xy}\sum_{k=0}^{\infty} p^{(k)}(y)\frac{(-xy(y+1))^k}{(\alpha+1)\cdots(\alpha+k)k!}.$$

Since $\{((\alpha+1)\cdots(\alpha+k))^{-1}\}_{k=0}^{\infty}$ is a non-negative multiplier sequence as proved already by Laguerre [11], and as such preserves $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ when acting on x (see [6] and Lemma 3.7 of [4]), it is enough to prove that

$$\sum_{k=0}^{n} p^{(k)}(y)\frac{(-yx(y+1))^k}{k!}$$

is a stable polynomial in two variables. Now

$$\begin{aligned}\sum_{k=0}^{n} p^{(k)}(y)\frac{(-xy(y+1))^k}{k!} &= p(y - xy(y+1)) \\ &= \prod_{j=0}^{n}(\theta_j + y - xy(y+1))\end{aligned}$$

where $0 \le \theta_j \le 1$. Observe that $y + \theta_j \ll y(y+1)$ for all $0 \le \theta_j \le 1$ and thus, by e.g. [5, Lemma 2.8], it follows that each factor is stable. This finishes the proof of Theorem 1.1.

3. Hermite multiplier sequences

We will now apply our methods to give a short proof of the characterization of Hermite multiplier sequences achieved by Piotrowski [13]. The Hermite polynomials, $\mathscr{H} = \{H_n(x)\}_{n=0}^{\infty}$, may be defined by the generating function

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n, \tag{3.1}$$

see [16]. Since Hermite polynomials are even or odd it is easy to see that $\{\lambda_n\}_{n=0}^{\infty}$ is an $\mathscr{H}$-multiplier sequence if and only if $\{(-1)^n\lambda_n\}_{n=0}^{\infty}$ is an $\mathscr{H}$-multiplier sequence. It is also plain to see that any trivial sequence is an $\mathscr{H}$-multiplier sequence, and that all $\mathscr{H}$-multiplier sequences are multiplier sequences (see [13, Theorem 158]). Since the entries of multiplier sequences either have the same sign or alternate in sign (by Lemma 2.4 (1)) it remains to characterize non-negative and non-trivial Hermite multiplier sequences. In [13] a generalization of Pólya's curve theorem led to the following characterization, which we will now re-prove:

Theorem 3.1 (Piotrowski, [13])**.** *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a non-trivial sequence of non-negative numbers. Then $\{\lambda_n\}_{n=0}^{\infty}$ is a Hermite multiplier sequence if and only if it is a (classical) multiplier sequence with $\lambda_n \le \lambda_{n+1}$ for all $n \ge 0$.*

Let $\{\lambda_n\}_{n=0}^{\infty}$ be a non-trivial and non-negative classical multiplier sequence and let T be the corresponding operator. Note that (3.1) implies

$$e^{-xy} = e^{y^2/4}\sum_{k=0}^{\infty}\frac{H_k(x)}{k!}\left(\frac{-y}{2}\right)^k,$$

and thus

$$G_T(x,y) = T(e^{-xy}) = e^{y^2/4} \sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!}$$

is the symbol of T. By Theorem 2.1 we want to determine when $G_T(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ or $G_T(-x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$. First let us prove that $G_T(-x,y)$ is never in $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$. Suppose that $G_T(-x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ and let M be the first index for which $\lambda_M \neq 0$. Then, since $e^{-y^2/4} \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$,

$$y^{-M} e^{-y^2/4} G_T(-x,y) = \frac{\lambda_M H_M(x)}{2^M M!} + \frac{\lambda_{M+1} H_{M+1}(x)}{2^{M+1}(M+1)!} y + \cdots \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R}). \tag{3.2}$$

Since $\{\lambda_n\}_{n=0}^{\infty}$ is nonnegative, Lemma 2.4 (2) implies $\lambda_M, \lambda_{M+1} > 0$, and as in the previous section we may apply the multiplier sequence $\{1, 1, 0, \ldots\}$ (acting on y) to (3.2) and conclude

$$\frac{\lambda_M H_M(x)}{2^M M!} + \frac{\lambda_{M+1} H_{M+1}(x)}{2^{M+1}(M+1)!} y \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R}),$$

and as before this implies $H_{M+1}(x) \ll H_M(x)$ which does not hold (although $H_M(x) \ll H_{M+1}(x)$ is a standard fact about orthogonal polynomials). Hence we have arrived at a contradiction.

It remains to find necessary and sufficient conditions for $G_T(x,y)$ to be in the Laguerre–Pólya class. Now

$$\sum_{k=0}^{\infty} \frac{H_k(x)(-y)^k}{2^k k!} = e^{-xy} e^{-y^2/4} \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R}). \tag{3.3}$$

Hence for a non-negative multiplier sequence $\{\lambda_n\}_{n=0}^{\infty}$,

$$\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R}).$$

Recall the representation (2.1) of entire functions in $\mathscr{L}\text{-}\mathscr{P}_1(\mathbb{R})$. A similar representation holds for $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$, see [12, p. 370]:

Theorem 3.2. *If $f(x,y)$ is an entire function of two variables, then f is in $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{C})$ if and only if f has the representation*

$$f(x,y) = e^{-ax^2 - by^2} f_1(x,y),$$

where a and b are non-negative numbers and f_1 is in $\mathscr{L}\text{-}\mathscr{P}_2(\mathbb{C})$ and of order at most one in each of its variables under the condition that the other variable is fixed in the open upper half-plane.

Thus we may write

$$\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} = e^{-ax^2 - by^2} g(x,y) \tag{3.4}$$

for some entire function $g(x,y) \in \mathscr{L}\text{-}\mathscr{P}_2(\mathbb{R})$ of order at most 1 in each variable under the condition that the other variable is fixed in the open upper half-plane. Hence

$$G_T(x,y) = e^{y^2/4} \sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} = e^{-ax^2 - (b-1/4)y^2} g(x,y). \tag{3.5}$$

In light of Theorem 3.2 our task has reduced to establishing when $b \geq 1/4$.

Recall that the *order* ρ, and *type* σ of an entire function $f(x)$ may be defined as:

$$\rho := \overline{\lim_{r\to\infty}} \frac{\ln\ln M(r)}{\ln r} \qquad \text{and} \qquad \sigma := \overline{\lim_{r\to\infty}} \frac{\ln M(r)}{r^\rho},$$

where $M(r) := \max_{|z|=r} |f(z)|$. In terms of its Taylor coefficients, $\{c_n\}_{n=0}^\infty$, the order and type of f are given by

$$\rho = \overline{\lim_{n\to\infty}} \frac{n \ln n}{\ln \frac{1}{|c_n|}} \qquad \text{and} \qquad (\sigma e \rho)^{1/\rho} = \overline{\lim_{n\to\infty}} n^{1/\rho} |c_n|^{1/n}, \tag{3.6}$$

see e.g. [12, p. 4].

Lemma 3.3. *Let $\{\lambda_n\}_{n=0}^\infty$ be a non-negative multiplier sequence with exponential generating function given by (2.3), and let $\sum_{n=0}^\infty c_n x^n$ be an entire function in $\mathscr{L}$-$\mathscr{P}_1(\mathbb{C})$ of order 2 and type c. Then*

$$\sum_{n=0}^\infty \lambda_n c_n z^n = \exp(-cs^2x^2) f(x) \tag{3.7}$$

where $f(x)$ has order at most one.

Proof. By continuity we may assume that $s > 0$. Then, by (3.6), the order of (3.7) is 2. Let σ be the type of the left hand side of (3.7). By (3.6) again,

$$(\sigma e 2)^{1/2} = \overline{\lim_{n\to\infty}} n^{1/2} (\lambda_n |c_n|)^{1/n} = \overline{\lim_{n\to\infty}} n^{1/2} \left(\frac{\lambda_n}{n!}\right)^{1/n} (n!)^{1/n} |c_n|^{1/n}.$$

Since $(n!)^{1/n} \sim ne^{-1}$,

$$(\sigma e 2)^{1/2} = e^{-1} \overline{\lim_{n\to\infty}} n^{1/2} |c_n|^{1/n} n \left(\frac{\lambda_n}{n!}\right)^{1/n} = e^{-1}(ce2)^{1/2} se,$$

that is, $\sigma = cs^2$. $\square$

We may now establish when $b \geq 1/4$ in (3.4) and thus finish our proof of Theorem 3.1. Since the order and type with respect to y of (3.3) is 2 and $1/4$, it follows by Lemma 3.3 that $b = s^2/4$. Theorem 3.1 now follows from the following lemma of Craven and Csordas:

Lemma 3.4 (Lemma 2.2, [8])**.** *Let $\{\lambda_n\}_{n=0}^\infty$ be a non-negative multiplier sequence with exponential generating function given by (2.3). Then $s \geq 1$ if and only if $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$.*

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